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On the Jackson–Müntz Theorem

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1. INTRODUCTION

A well known result of Müntz asserts that for an increasing sequence of nonnegative real numbers $\{p_i\}_{i=0}^{\infty}$ with $\lim_{i\to\infty} p_i = \infty$, a necessary and sufficient condition for the finite linear combinations of x^{p_i} (the "polynomials" $\sum_{i=0}^{n} a_i x^{p_i}$) to be dense in $C[0, 1]$ is that $p_0 = 0$ and $\sum_{i=1}^{\infty} 1/p_i = \infty$. We shall assume these conditions throughout this note.

We shall be concerned with the problem of how well can a given function $f(x) \in C[0, 1]$ be approximated, in the sup norm, by "polynomials." In other words, we look for an estimate on

$$E_n(f; p_i) = \min_{a_i} \max_{0 \leqslant x \leqslant 1} \left| f(x) - \sum_{i=0}^{n} a_i x^{p_i} \right|.$$

For the sequence $p_i = i$ $(i \geqslant 0)$ this problem was completely solved by Jackson and the estimates were given in terms of the modulus of continuity of the function or its kth derivative if the latter exists and is continuous in [0, 1]. An attempt to generalize this to sequences satisfying Müntz's conditions and $p_{i+1} - p_i \geqslant 2$ was made by Newman [3] and, more recently, in a series of papers [4, 5, 6] von Golitschek has given estimates on $E_n(f; p_i)$ for sequences satisfying Müntz's conditions and the inequalities

$$A(p_n)^{\delta} \leqslant \exp\left(\sum_{i=1}^{n} 1/p_i\right) \leqslant B(p_n)^{\Delta}, \qquad n = 1, 2,..., \tag{1}$$

for some positive constants A, B, δ, Δ $(\geqslant \delta)$.

Our results include those of von Golitschek and unify the different cases he has.

2. Main Results

As was mentioned above, we assume

$$0 = p_0 < p_1 < \cdots < p_n < \cdots \to \infty; \qquad \sum_{i=1}^{\infty} 1/p_i = \infty. \tag{2}$$

Denote

$$\sigma_{nj} = \max_{0 \leqslant r \leqslant j} \left\{ 1/p_r \exp \left[-2 \sum_{i=r+1}^{n} 1/p_i \right] \right\}, \tag{3}$$

then $\sigma_{nn} \geqslant 1/p_n$, but it follows immediately by (2) that σ_{nn} tends to 0 as $n \to \infty$.

Our first result concerns the function $f(x) = x^a$ for which our estimates are given by means of σ_{nr}, $p_r \leqslant a < p_{r+1}$. For the general case we have estimates given by means of the bigger quantity σ_{nn}.

Theorem 1. *Let a be a positive number, different from all p_i, $i \geqslant 0$; and let n satisfy $p_n > a$. Then for r with $p_r \leqslant a < p_{r+1}$ we have*

$$E_n(x^a; p_i) \leqslant a^a \sigma_{nr}^a \tag{4}$$

and so

$$E_n(x^a; p_i) = 0(\sigma_{nr}^a), \qquad as \quad n \to \infty.$$

Theorem 2. *Let $f(x)$ be k-times continuously differentiable in* $[0, 1]$ $(k \geqslant 0)$. *Then there exist constants C_k and K_k such that for all n with $p_n \geqslant 2k + 1$,*

$$E_n(f; p_i) \leqslant C_k \sigma_{nn}^k \omega(f^{(k)}; \sigma_{nn}) + K_k \sigma_{ns}^\kappa, \tag{5}$$

where $\omega(f^{(k)}; \cdot)$ is the modulus of continuity of $f^{(k)}$, $p_s \leqslant k < p_{s+1}$ and

$$\kappa = \min\{j \mid 1 \leqslant j \leqslant k, j \notin \{p_i\}, f^{(j)}(0) \neq 0\}$$

(κ is ∞ if either $k = 0$ or the above set is empty so that in this case $\sigma_{ns}^\kappa = 0$.)

In the proof we need two lemmas of von Golitschek (Lemma 2 and Satz 4 of [6]).

Lemma A. *Let a be a positive real number. Then*

$$E_n(x^a; p_i) \leqslant \prod_{i=1}^{n} \frac{|a - p_i|}{a + p_i}$$

LEMMA B. *Let f be k-times continuously differentiable in* $[0, 1]$ $(k \geqslant 0)$ *and let $n > 2k$ and $\lambda \geqslant 1$. Then there exists a polynomial $T_n(x) = \sum_{i=0}^{n} b_{ni}x^i$ such that we have*

$$|f^{(j)}(x) - T_n^{(j)}(x)| \leqslant A_k(\lambda/n)^{k-j}\, \omega(f^{(k)}; \lambda/n), \quad j = 0,..., k, \quad 0 \leqslant x \leqslant 1 \quad (6)$$

and

$$|b_{ni}| \leqslant A_k(n/\lambda)^{i-k}\, \omega(f^{(k)}; \lambda/n)/i!, \qquad i = k+1,..., n. \quad (7)$$

Proof of Theorem 1. It is known that for $0 \leqslant x \leqslant 1$ we have $(1 - x)/(1 + x) \leqslant e^{-2x}$. Hence, by Lemma A for n such that $p_n > a$

$$E_n(x^a; p_i) \leqslant \exp\left[-2 \sum_{i=r+1}^{n} a/p_i\right]$$

where $p_r \leqslant a < p_{r+1}$. This implies

$$E_n(x^a; p_i) \leqslant a^a \left\{1/p_r \exp\left[-2 \sum_{i=r+1}^{n} 1/p_i\right]\right\}^a$$
$$\leqslant a^a \sigma_{nr}^a .$$

This completes the proof of (4).

Proof of Theorem 2. Let $p_n \geqslant 2k + 1$ and set $m = [p_n]$ and $\lambda = 2em\sigma_{nn}$ so $\lambda \geqslant 1$. By Lemma B, there exists a polynomial $T_m(x) = \sum_{i=0}^{m} b_{mi}x^i$ such that

$$|f(x) - T_m(x)| \leqslant A_k(\lambda/m)^k\, \omega(f^{(k)}; \lambda/m). \quad (8)$$

Now if $j \notin \{p_i\}$ let $Q_{nj} = \sum_{i=0}^{n} a_{ji}x^{p_i}$ be the "polynomial" of best approximation to x^j and define

$$\tilde{T}_n(x) = \sum_{\substack{j=0 \\ j\in\{p_i\}}}^{m} b_{mj}x^j + \sum_{\substack{j=1 \\ j\notin\{p_i\}}}^{m} b_{mj}Q_{nj}(x).$$

Then

$$|f(x) - \tilde{T}_n(x)| \leqslant |f(x) - T_m(x)| + \sum_{\substack{j=1 \\ j\notin\{p_i\}}}^{m} |b_{mj}|\, |x^j - Q_{nj}(x)|. \quad (9)$$

For $1 \leqslant j \leqslant k$ it follows by (6) that

$$|b_{mj}| \leqslant f^{(j)}(0)/j! + A_k(\lambda/m)^{k-j}\, \omega(f^{(k)}; \lambda/m).$$

Also by Theorem 1,

$$\begin{aligned} |x^j - Q_{nj}(x)| &\leqslant E_n(x^j; p_i) \\ &\leqslant j^j \sigma_{ns}^j \end{aligned}$$

and so since $\sigma_{nn} \geqslant \sigma_{ns}$,

$$|b_{mj}|\,|x^j - Q_{nj}(x)| \leqslant f^{(j)}(0)\, e^j \sigma_{ns}^j + B_k \sigma_{nn}^k \omega(f^{(k)}; 2e\sigma_{nn}), \tag{10}$$

where $B_k = A_k(2e)^k k^k$.

For $k + 1 \leqslant j \leqslant m$ it follows by (7) and Theorem 1 that

$$\begin{aligned} |b_{mj}|\,|x^j - Q_{nj}(x)| &\leqslant A_k(m/\lambda)^{j-k}\, \omega(f^{(k)}; \lambda/m)\, e^j \sigma_{nn}^j \\ &\leqslant B_k \sigma_{nn}^k \omega(f^{(k)}; 2e\sigma_{nn})\, 2^{-j}. \end{aligned} \tag{11}$$

We have

$$\omega(f^{(k)}; 2e\sigma_{nn}) \leqslant (2e + 1)\, \omega(f^{(k)}; \sigma_{nn})$$

and so combining (8)–(11), (5) follows. This completes the proof.

3. Applications

It is readily seen that if $\{p_i\}$ satisfies (1), then for any fixed $k \geqslant 0$

$$\sigma_{nk} = 0(p_n^{-\delta})$$

and

$$\sigma_{nn} = \begin{cases} 0(p_n^{-2\delta}) & \Delta < \frac{1}{2}, \\ 0(p_n^{-\delta/\Delta}) & \Delta \geqslant \frac{1}{2}; \end{cases}$$

hence, our Theorems 1 and 2 reduce in this case to von Golitschek's Satz 4 and 5 [6], respectively.

If $p_i \geqslant i\lambda$ for some $\lambda > 0$, then it is readily seen that

$$\sigma_{nn} = \begin{cases} \exp\left[-2 \sum_{i=1}^{n} 1/p_i\right], & \lambda \geqslant 2, \\ \exp\left[-\lambda \sum_{i=1}^{n} 1/p_i\right], & 0 < \lambda < 2. \end{cases}$$

If, for example, $p_i = (i + 1)\log(i + 1)$, then both σ_{nk} (k fixed) and σ_{nn} are $O(1/(\log n)^2)$, as $n \to \infty$.

It is interesting to compare our estimates here with the known estimates on the rate of convergence of the generalized Bernstein polynomials [1]. It was shown in [1] Theorem 9 that if $p_1 = 1$, then the nth generalized Bernstein polynomial of $f(x) \in C[0, 1]$ approximates the function at least at the order of $\omega(f; \rho_n^{1/2})$, where

$$\rho_n = \max_{1 \leqslant r \leqslant n} \left\{ 1/p_r \exp \left[-(2 - \epsilon) \sum_{i=r+1}^{n} 1/p_i \right] \right\},$$

$\epsilon > 0$ is arbitrary and can be done away with in many cases, for instance if $p_{i+1} - p_i \geqslant a > 0$, $i \geqslant 0$. (See remarks in [1] at the top of p. 407.) In any case, ρ_n resembles σ_{nn} very much and we can say roughly that while, in general $\omega(f; \sigma_{nn})$ is the order of approximation of "polynomials" to $f(x)$, the generalized Bernstein polynomials allows order of approximation of $\omega(f; \sigma_{nn}^{1/2})$ —a familiar result for the ordinary Bernstein polynomials.

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